A characterization of normal 3-pseudomanifolds with $g_2 \leq 4$

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Abstract

For a d-dimensional simplicial complex K, the f-vector is defined as $(f_{-1}, f_0, f_1, \ldots, f_d)$, where $f_{-1} = 1$ and f_i denotes the number of i-dimensional faces of K. The f-vector plays a fundamental role in simplicial topology and combinatorial geometry. In dimension three, an important invariant derived from the f-vector is

$$g_2(K) = f_1 - 4f_0 + 10.$$

Walkup [6] proved that $g_2(K) \geq 0$ for any closed connected triangulation of a 3-manifold and that $g_2(K) = 0$ if and only if K is a stacked sphere. Subsequently, Barnette [2] established the non-negativity of g_2 for boundary complexes of simplicial polytopes. Using rigidity theory introduced by Kalai [4] and results of Fogelsanger [3], it is now known that the inequality $g_2(K) \geq 0$ holds for all normal d-pseudomanifolds with $d \geq 3$.

Several structural results are known for complexes with small values of g_2. The cases $g_2 = 0, 1,$ and 2 were characterized in earlier works of Kalai, Nevo–Novinsky [5] , and Zheng [7], where in each case the complexes are polytopal spheres. The case $g_2 = 3$ for normal 3-pseudomanifolds was studied by Basak and Swartz [1]. These developments naturally lead to the question posed in [1]:

Question. Is there a 3-dimensional normal pseudomanifold Δ such that the minimum value of g_2 over all triangulations homeomorphic to Δ is 4 or 5?

In this work we show that the answer to this question is negative and obtain a complete structural description of normal 3-pseudomanifolds K satisfying $g_2(K) \leq 4$. We first prove that such complexes have very restricted singularity structure.

Theorem 1. *If K is a normal 3-pseudomanifold with $g_2(K) \leq 4$, then K has either no singular vertices or exactly two singular vertices.*

When there are no singular vertices, K is a combinatorial 3-manifold. Using known results of Walkup, we obtain the following characterization.

Theorem 2. *If K is a normal 3-pseudomanifold with $g_2(K) \leq 9$ and no singular vertices, then K is a triangulation of the 3-sphere and can be obtained from boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, and edge expansions.*

We then analyze the case when exactly two singular vertices occur.

Theorem 3. *If K is a normal 3-pseudomanifold with $g_2(K) \leq 4$ and exactly two singular vertices, then K can be obtained from boundary complexes of 4-simplices by a sequence of operations consisting of connected sums, edge expansions, and an edge folding.*

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Combining these results, we obtain the following topological classification.

Theorem 4. *If K is a normal 3-pseudomanifold with $g_2(K) \leq 4$, then K is a triangulation of either the 3-sphere or the suspension of $\mathbb{RP}^2$.*

A detailed exposition of these results can be found in the preprint available at

`https://arxiv.org/pdf/2202.06638`.

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